

Examples – Continuous Compounding of Interest

1. Suppose that in year 0, one cent was invested in an account earning 1% interest compounded continuously (i.e., $r = 0.01$). How much will it be worth 2110 years later?

The amount an account has grown to after t years is given by $A(t) = A_0e^{rt}$, which in this case becomes:

$$A(t) = 0.01e^{0.01t}$$

So after 2110 years, the account will have grown to

$$\begin{aligned} A(2110) &= 0.01e^{(0.01)2110} = 0.01e^{21.1} \\ &= (0.01)(1457516796.05142392) \\ &= 14575167.9605142392 \end{aligned}$$

$$\boxed{\$14,575,167.96}.$$

2. How much would you have to invest in an account earning 8% interest compounded continuously (i.e., $r = 0.08$), for it to be worth one million dollars in 30 years?

The amount an account has grown to after t years is given by $A(t) = A_0e^{rt}$. Thus,

$$A(30) = A_0e^{0.08(30)} = 1000000 \Rightarrow A_0 = \frac{1000000}{e^{2.4}} \cong 90717.95$$

At this rate, $\boxed{\$90,717.95}$ will grow to one million dollars in 30 years.

3. At what interest rate (compounded continuously) would \$3,000 grow to \$300,000 in 25 years?

The amount an account has grown to after t years is given by $A(t) = A_0e^{rt}$. Thus,

$$A(25) = 3000e^{r(25)} = 300000 \Rightarrow e^{25r} = \frac{300000}{3000} = 100$$

Taking the natural log of both sides of this equation then yields:

$$25r = \ln 100 \Rightarrow r = \frac{\ln 100}{25} \cong 0.18420680743952365$$

The interest rate would have to be about $\boxed{18.42\%}$.

4. What is an investment's doubling time, to the nearest ten thousandths of a year, if it earns 5% interest compounded continuously?

The amount an account has grown to after t years is given by $A(t) = A_0e^{rt}$. Thus,

$$A(t) = A_0e^{0.05t} = 2A_0 \Rightarrow e^{0.05t} = 2$$

Taking the natural log of both sides of this equation then yields:

$$0.05t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.05} \cong 13.862943611198906$$

At 5% interest, compounded continuously, it will take about 13.8629 years for an investment to double.

Examples – Unconstrained Population Growth

5. The population of Mexico was 100,349,766 in the year 2000, and was 109,955,400 in the year 2008. Predict the population of Mexico in the year 2020.

Unconstrained population growth is described by the equation $A(t) = A_0e^{rt}$.

And so, the Population of Mexico, t years after the year 2000, is given by:

$$A(t) = 100349766 e^{rt}$$

We can use the given information to find the growth rate, r : Since 2008 is 8 years after the year 2000,

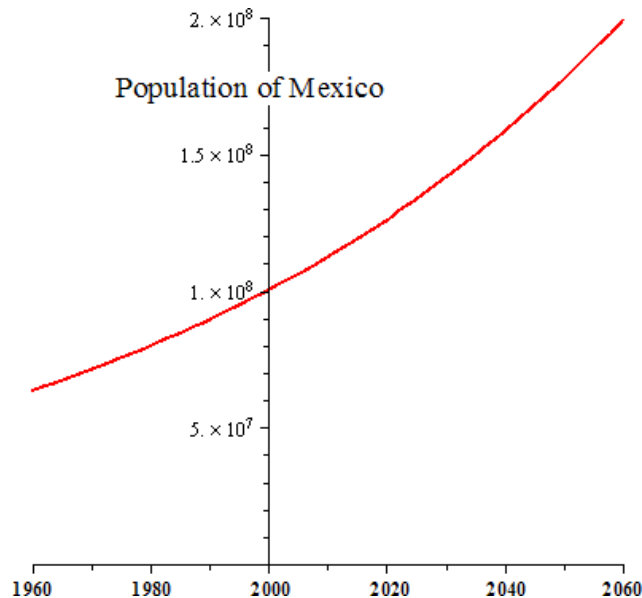
$$A(8) = 100349766 e^{r \cdot 8} = 109955400 \Rightarrow e^{8r} = \frac{109955400}{100349766}$$

Taking the natural log of both sides of this equation yields:

$$8r = \ln \frac{109955400}{100349766} \Rightarrow r = \frac{\ln \frac{109955400}{100349766}}{8} \cong 0.01142663570337241$$

So our equation is as before, $A(t) = 100349766 e^{rt}$, but now we know exactly what r equals. Now we can make our prediction:

$$A(20) = 100349766 e^{r \cdot 20} = 100349766(1.25675463801540319) \cong \boxed{126,115,034}$$



6. Suppose that a population of bacteria doubles every 43 minutes.
If a bacteria population initially consists of 50 bacteria, how many will there be after 6 hours?

The number of bacteria in a population, after t minutes, is given by $A(t) = A_0 e^{rt}$, which in this case becomes

$$A(t) = 50e^{rt}$$

We can use the doubling time to find the growth rate, r :

$$A(43) = 50e^{r \cdot 43} = 100 \Rightarrow e^{43r} = \frac{100}{50} = 2$$

Taking the natural log of both sides of this equation yields:

$$43r = \ln 2 \Rightarrow r = \frac{\ln 2}{43} \cong 0.0161197018734871$$

I like to keep this as simply r in my calculations, until the very end when it's time to compute a final numerical answer to the problem. That way there's much less writing! ☺ So our equation is as before, $A(t) = 50e^{rt}$, but now we know exactly what r equals. Thus, after 6 hours (360 minutes):

$$A(360) = 50e^{r \cdot 360} = 50e^{0.0161197018734871 \cdot 360} = 50e^{5.8030926744553560}$$

$$= 50(331.3226501482628) \cong 16566.1325$$

After 6 hours there will be about 16,566 bacteria. Be sure to refrigerate that potato salad!

7. An initial population of 123 bacteria has grown to 321 bacteria after one hour. What is the doubling-time for this population, to the nearest second?

The number of bacteria in a population, after t minutes, is given by $A(t) = A_0e^{rt}$, so we have:

$$A(t) = 123e^{rt}$$

We can use the given information to find the growth rate, r :

$$A(60) = 123e^{r \cdot 60} = 321 \quad \Rightarrow \quad e^{60r} = \frac{321}{123} = \frac{107}{41}$$

Taking the natural log of both sides of this equation yields:

$$60r = \ln \frac{107}{41} \quad \Rightarrow \quad r = \frac{\ln \frac{107}{41}}{60} \cong 0.01598761279595997$$

So our equation is as before, $A(t) = 123e^{rt}$, but now we know exactly what r equals. Now we can solve for the doubling time:

$$A(t) = 123e^{rt} = 2 \cdot 123 \quad \Rightarrow \quad e^{rt} = \frac{2 \cdot 123}{123} = 2$$

Taking the natural log of both sides of this equation yields:

$$rt = \ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{r} \cong \frac{\ln 2}{0.01598761279595997} \cong 43.35526$$

And so, the doubling time of the bacterial population is about 43.35526 minutes, which is about 43 minutes and 21.3158665 seconds \cong 43 minutes 21 seconds.

Examples – Radioactive Decay (In this case $r < 0$)

8. Starting with 9.1 grams of a radioactive substance, one finds that 5.3 grams remain after 6 days. What is the half-life of this substance, to the nearest second?

Radioactive decay is described by the equation $A(t) = A_0 e^{rt}$. Substituting the initial amount (1 kilogram), this equation becomes

$$A(t) = 9.1 e^{rt},$$

where the time t is measured in days. We can use the given information to find r :

$$A(6) = 9.1 e^{r \cdot 6} = 5.3 \Rightarrow e^{6r} = \frac{5.3}{9.1}$$

Taking the natural log of both sides of this equation yields:

$$6r = \ln \frac{5.3}{9.1} = \ln \frac{53}{91} \Rightarrow r = \frac{\ln \frac{53}{91}}{6} \cong -0.0900945988274547$$

So our equation is as before, $A(t) = 9.1 e^{rt}$, but now we know exactly what r equals. If t is the half-life of the substance, then:

$$A(t) = 9.1 e^{rt} = \frac{9.1}{2} = 4.55 \Rightarrow e^{rt} = \frac{4.55}{9.1}$$

Taking the natural log of both sides of this equation, we can solve for t :

$$rt = \ln \frac{4.55}{9.1} \Rightarrow t = \frac{\ln \frac{4.55}{9.1}}{r} = \frac{\ln \frac{4.55}{9.1}}{\left(\frac{\ln \frac{53}{91}}{6} \right)} \cong 7.69354866530$$

The half life 7.6935486653 days, = 7 days 41 minutes 36.775 seconds,

or about 7 days 48 minutes 14 seconds .

8. Fluoro-18-deoxyglucose is used in PET scanners to measure glucose metabolism in the brain. The F18 involved in its production has a half-life of 110 minutes. Starting with a 1 kilogram sample of this substance, how much will remain after 24 hours have elapsed?

Radioactive decay is described by the equation $A(t) = A_0 e^{rt}$. Substituting the initial amount (1 kilogram), this equation becomes

$$A(t) = e^{rt},$$

where the time t is measured in minutes. We can use the given half-life to find the growth rate, r :

$$A(110) = e^{r \cdot 110} = \frac{1}{2} \Rightarrow e^{110r} = \frac{1}{2}$$

Taking the natural log of both sides of this equation yields:

$$110r = \ln \frac{1}{2} \Rightarrow r = \frac{\ln 0.5}{110} \cong -0.0063013380050904119$$

So our equation is as before, $A(t) = e^{rt}$, but now we know exactly what r equals. Now we can solve for how much remains after 24 hours = 1440 minutes:

$$\begin{aligned} A(1440) &= e^{r \cdot 1440} \cong e^{(-0.006301338005) \cdot 1440} \cong e^{-9.07392672733} \\ &\cong 0.00011461558968 \end{aligned}$$

After 24 hours, only about 0.1146 grams remain. Just a little over a tenth of a gram!
That's not a very good shelf-life! ☺

9. The half-life of carbon-14 is about 5730 years. A gram of charcoal from the cave paintings in France gives 0.97 disintegrations per minute, while a gram of living wood gives 6.68 disintegrations per minute. Find the age of those Lascaux paintings.

(Note: Strang missed this one in his text, both for using the wrong C-14 half-life, and a typo too.) First we'll use the given half-life to find the growth rate, r :

Radioactive decay is described by the equation $A(t) = A_0 e^{rt}$. Solving for r , we have: where the time t is measured in years. After 5730 years, there will be half as much, $\frac{1}{2}$:

$$A(5730) = A_0 e^{r \cdot 5730} = \frac{A_0}{2} \Rightarrow e^{5730r} = \frac{1}{2}$$

Taking the natural log of both sides of this equation yields:

$$5730r = \ln \frac{1}{2} \Rightarrow r = \frac{\ln 0.5}{5730} \cong -0.00012096809433855939$$

So now we know r for carbon-14. The situation described in the problem says initially the radioactivity, A_0 , was 6.68 disintegrations per minute, and after t years this has decayed to only 0.97 disintegrations per minute. We can model the amount of radioactivity at time t via

$$A(t) = 6.68e^{rt} = 0.97 \Rightarrow e^{rt} = \frac{0.97}{6.68}$$

Taking the natural log of both sides of this equation, we can solve for t :

$$rt = \ln \frac{0.97}{6.68} \Rightarrow t = \frac{\ln \frac{0.97}{6.68}}{r} = \frac{\ln \frac{0.97}{6.68}}{\left(\frac{\ln 0.5}{5730}\right)} \cong 15951.125$$

The sample is about 15,951 years old.

Newton's Law of Cooling

If T_a is the ambient (background, constant) temperature, and T_0 is the initial temperature of an object, then the temperature at time t is given by the function

$$T(t) = T_a - (T_a - T_0)e^{-rt}$$

The cooling rate r is a constant that depends on characteristics of the cooling (or warming) object. Notice that if $T_a - T_0 > 0$ then the object is warming, and if $T_a - T_0 < 0$ then the object is cooling.

Example – Newton's Law of Cooling

10. Suppose that CSI finds a body in a hotel room, and measures its temperature to be 70.4 degrees at 11am, and 67.5 degrees one hour later. If the thermostat in the room was set to 62 degrees, what was the time of death?

Newton's law of cooling implies that the temperature of the body t hours after death is given by:

$$T(t) = 62 - (62 - 98.6)e^{-rt} \Rightarrow T(t) = 62 + 36.6e^{-rt}$$

Let t represent the time elapsed since death at 11am. Then:

$$T(t) = 62 + 36.6e^{-rt} = 70.4 \Rightarrow 36.6e^{-rt} = 8.4 \Rightarrow e^{-rt} = \frac{8.4}{36.6},$$

Taking logs, we get that $-rt = \ln \frac{8.4}{36.6}$.

One hour later, at time $t+1$, we have:

$$T(t+1) = 62 + 36.6e^{-r(t+1)} = 67.5 \Rightarrow 36.6e^{-r(t+1)} = 5.5 \Rightarrow e^{-r(t+1)} = \frac{5.5}{36.6}$$

Taking logs, we get that $-r(t+1) = \ln \frac{5.5}{36.6} \Rightarrow -rt - r = \ln \frac{5.5}{36.6}$. Substituting the value we just calculated for $-rt$ above, $-rt = \ln \frac{8.4}{36.6}$, we get:

$$\begin{aligned} -rt - r &= \ln \frac{5.5}{36.6} \Rightarrow \ln \frac{8.4}{36.6} - r = \ln \frac{5.5}{36.6} \Rightarrow r = \ln \frac{8.4}{36.6} - \ln \frac{5.5}{36.6} \\ &\Rightarrow r = \ln \left(\frac{\frac{8.4}{36.6}}{\frac{5.5}{36.6}} \right) = \ln \left(\frac{8.4}{36.6} \cdot \frac{36.6}{5.5} \right) = \ln \left(\frac{8.4}{5.5} \right) = \ln \left(\frac{84}{55} \right) \end{aligned}$$

So now we know that $T(t) = 62 + 36.6e^{-rt}$, but now we know exactly what r is, so we can now find t , the time elapsed since death:

$$T(t) = 62 + 36.6e^{-rt} = 70.4 \Rightarrow 36.6e^{-rt} = 8.4 \Rightarrow e^{-rt} = \frac{8.4}{36.6}$$

Taking logs, we get:

$$-rt = \ln \frac{8.4}{36.6} \Rightarrow t = \frac{\ln \frac{8.4}{36.6}}{-r} = \frac{\ln \left(\frac{84}{366} \right)}{-\ln \left(\frac{84}{55} \right)} \cong 3.47549819$$

So the time of death occurred 3 hours and 28.53 minutes prior to 11am, or at about 7:31 am.

