Section 2.5 # 72

Find the maximum value of the function $f(x) = 3 + 4x^2 - x^4$. (Hint: Let $t = x^2$.)

OK, let's take a hint! The graph of this function is the set of points (x, y) satisfying the equation $y = 3 + 4x^2 - x^4 \quad \Rightarrow \quad y = 3 + 4t - t^2$, but notice in that later equation that we restrict the domain to only non-negative real values of t.

Let's complete the square:

$$y = -t^2 + 4t + 3 = -\left(t^2 - 4t - 3\right)$$

$$y = -\left(t^2 - 4t + \boxed{4} - 3 - \boxed{4}\right)$$

$$y = -\left(\left(t - \boxed{2}\right)^2 - 7\right) = -\left(t-2\right)^2 + 7$$

$$y = 7 - \left(t-2\right)^2 \quad \Rightarrow \quad \boxed{y = 7 - \left(x^2 - 2\right)^2}$$

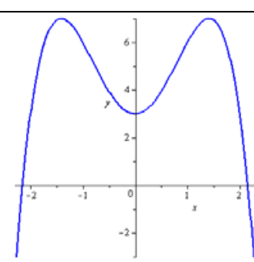
Notice that in this form, it's clear that this function attains its maximum of $y = 7$ when $x = \pm\sqrt{2}$. (This is because then you're subtracting 0 from 7. Other values of x result in subtracting a positive value from the 7, thus diminishing it.)

Notice that this function is an even function, which is thus symmetric about the *y*-axis. To see that *f* is even (this just means that $f(-x) = f(x)$ for all *x*):

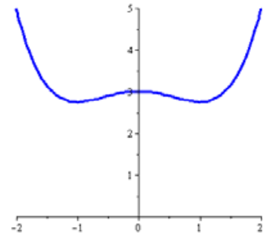
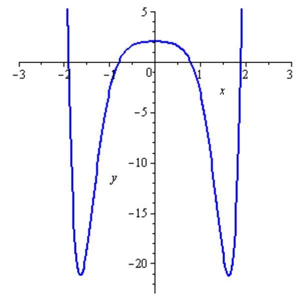
$$f(x) = 3 + 4x^2 - x^4$$

$$f(-x) = 3 + 4(-x)^2 - (-x)^4 = 3 + 4x^2 - x^4 = f(x)$$

Being symmetric about the *y*-axis guarantees that the *y*-intercept has a horizontal tangent line (think about symmetry about the *y*-axis . . . I've included several graphs of even functions at the end of this problem solution). Thus, there must be a local minimum at the *y*-intercept (this occurs when *x*=0): $f(0) = 3 + 4 \cdot 0^2 - 0^4 = 3$. Below is the graph of this fourth degree trinomial function, $f(x) = 3 + 4x^2 - x^4$:

Notice how there's a local minimum at the *y*-intercept. There's always a local maximum or minimum at the *y*-intercept of an even function (the ones defined at *x*=0), as you can see in the following examples.	

Some other even polynomial functions and their graphs:

$g(x)=\frac{x^4}{4}-\frac{x^2}{2}+3$	
$h(x)=x^8-3x^6-2x^4-x^2+2$	
$C(x)=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+\frac{x^{12}}{479001600}-\frac{x^{14}}{87178291200}+\frac{x^{16}}{20922789888000}-\frac{x^{18}}{6402373705728000}$