Precalculus Chapter 2 Quiz Part II (sections 5 – 8)

Solutions to problems # 9, 14, 18, 19, 21-23, 28, 29, 33, 35, 36, 38, 39.

---

9. Find the domain and range of the function $f(x) = x^2 - 12x + 2$.

The domain of f is all real numbers, $\boxed{(-\infty,\infty)}$.

To find the range of f, first realize that the graph of f is an upward opening parabola, "U".
Thus its range is all real numbers greater than or equal to the *y*-coordinate of the parabola's vertex, which is the minimum possible output of function f. So we need to find the vertex of this parabola.

The *x*-coordinate of the vertex $= \dfrac{-b}{2a} = \dfrac{-(-12)}{2\cdot 1} = 6$, and the *y*-coordinate of the vertex $f(6) = 6^2 - 12\cdot 6 + 2 = -34$.

Thus the range of f is $\boxed{[-34,\infty)}$.

---

14. Find the maximum value of the function $f(x) = -x^4 + 6x^2 + 5$. (Hint: Let $t = x^2$.)

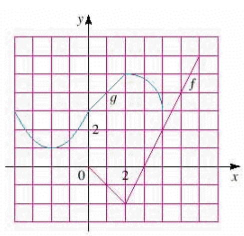
The graph of this function is the set of points (x, y) satisfying the equation
$y = -x^4 + 6x^2 + 5 \quad \Rightarrow \quad y = -t^2 + 6t + 5$, but notice in that later equation that we restrict the domain to only non-negative real values of t. Let's complete the square:

$$y = -t^2 + 6t + 5 = -(t^2 - 6t - 5)$$
$$y = -(t^2 - 6t + \boxed{9} - 5 - \boxed{9})$$
$$y = -\left((t - \boxed{3})^2 - 14\right) = -(t-3)^2 + 14$$
$$y = 14 - (t-3)^2 \quad \Rightarrow \quad \boxed{y = 14 - (x^2 - 3)^2}$$

Notice that in this form, it's clear that this function attains its maximum of $y = 14$ when $x = \pm\sqrt{3}$. (This is because then you're subtracting 0 from 14. Other values of x result in subtracting a positive value from the 14, thus diminishing it.)

---

18. Use the graphs of f and g to evaluate $g(f(5))$.



Solution:

From the graph, $f(5) = 4$.
From the graph, $g(4) = 3$.

Thus, $g(f(5)) = g(4) = \boxed{3}$

19. Find the domain of $g \circ f$, if $f(x) = x^2$ and $g(x) = \sqrt{x-20}$.

This composition has us first doing f and then doing g.

The domain of f is all real numbers. However, the domain of g is all real numbers greater than or equal to 20, (since then the radical is real). So we must exclude from the domain of $g \circ f$ any values output by f that are not in the domain of g. In other words, we must have

"Domain of $g \circ f$" =

$$\{x \mid x^2 \geq 20\} = \{x \mid x \geq \sqrt{20}\} \cup \{x \mid x \leq -\sqrt{20}\} = \boxed{\left(-\infty, -\sqrt{20}\right] \cup \left[\sqrt{20}, \infty\right)}$$

$$\text{Simplified} \rightarrow \boxed{\left(-\infty, -2\sqrt{5}\right] \cup \left[2\sqrt{5}, \infty\right)}$$

---

21. $F(x) = (x-10)^5 = (f \circ g)(x) = f(g(x))$

One can think of F as the function that first subtracts 10, and then raises the result to the fifth power, and so $\boxed{g(x) = x - 10, \quad f(x) = x^2}$ does the trick.

---

22. The graphs of the function $f(x) = m_1x + b_1$ and $q(x) = m_2x + b_2$ are lines with slopes m_1 and m_2 respectively. What is the slope of the graph of $f(q(x))$? (I did this one in class!)

$$f(x) = m_1 x + b_1 \quad \text{and} \quad q(x) = m_2 x + b_2 \quad \Rightarrow$$

$$(f \circ g)(x) = f(q(x)) = m_1 q(x) + b_1 = m_1(m_2 x + b_2) + b_1$$

$$f(q(x)) = m_1 m_2 x + m_1 b_2 + b_1$$

This is a line in slope intercept form, with slope = $\boxed{m_1 m_2}$, and y-intercept $m_1 b_2 + b_1$.

---

23. Suppose that $g(x) = 5x + 3$ and $h(x) = 25x^2 + 30x + 19$. Find a function f, such that $f(g(x)) = h(x)$.
(Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

If we square $5x + 3$, we get $25x^2 + 30x + 9$. This is just 10 short of being $h(x)$.
So if f is the function that first squares, and then adds 10, then our work is done! So, $\boxed{f(x) = x^2 + 10}$.

---

28. Assume f is a one-to-one function. If $f(x) = 3 - 6x$, find $f^{-1}(33)$. We start by finding f^{-1}:

$$y = 3 - 6x \quad \rightarrow \quad x = 3 - 6y$$

$$x - 3 = -6y$$

$$\frac{x-3}{-6} = y \quad \rightarrow \quad \boxed{f^{-1}(x) = \frac{3-x}{6}} \quad \Rightarrow \quad f^{-1}(33) = \frac{3-33}{6} = \frac{-30}{6} = \boxed{-5}$$

29. Assume g is a one-to-one function. If $g(x) = x^2 + 10x$ with $x \geq -5$, find $g^{-1}(4)$. We start by finding g^{-1}:

$$y = x^2 + 10x \quad \rightarrow \quad x = y^2 + 10y$$

$$0 = y^2 + 10y - x \quad \Rightarrow \quad y = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot (-x)}}{2 \cdot 1} = \frac{-10 \pm \sqrt{100 + 4x}}{2}$$

$$= \frac{-10 \pm \sqrt{4(25+x)}}{2} = \frac{-10 \pm 2\sqrt{25+x}}{2}$$

$$\boxed{g^{-1}(x) = -5 \pm \sqrt{25+x}}$$

We must decide which of these two possible inverse functions we need…
Notice that the point (0,0) is on the graph of g, and that x=0 is within the specified domain of g.
But then it must also be on the graph of g^{-1} (switch the coordinates and you still have (0,0)).
Thus we must select the +, since a negative would result in a function that doesn't go through (0,0).

Thus, $\boxed{g^{-1}(x) = -5 + \sqrt{25+x}} \quad \Rightarrow \quad g^{-1}(4) = -5 + \sqrt{25+4} = \boxed{-5+\sqrt{29}}$.

---

33. Find the inverse function of $f(x) = \dfrac{2-7x}{9-5x}$.

$$y = \frac{2-7x}{9-5x} \quad \rightarrow \quad x = \frac{2-7y}{9-5y}$$

$$x(9-5y) = 2-7y$$

$$9x - 5xy = 2 - 7y$$

$$7y - 5xy = 2 - 9x$$

$$y(7-5x) = 2-9x \quad \Rightarrow \quad y = \frac{2-9x}{7-5x}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{2-9x}{7-5x} = \boxed{\frac{9x-2}{5x-7}}$$

---

35. Find the inverse function of $f(x) = \left(7-x^3\right)^{\frac{1}{5}}$.

$$y = \left(7-x^3\right)^{\frac{1}{5}} \quad \rightarrow \quad x = \left(7-y^3\right)^{\frac{1}{5}}$$

$$x^5 = 7 - y^3$$

$$x^5 - 7 = -y^3$$

$$7 - x^5 = y^3$$

$$\left(7-x^5\right)^{\frac{1}{3}} = y \quad \Rightarrow \quad \boxed{f^{-1}(x) = \sqrt[3]{7-x^5}}$$

36. Find the inverse function of $f(x)=\sqrt{25-x^2}, \quad 0 \le x \le 5$.

$$y=\sqrt{25-x^2} \quad \rightarrow \quad x=\sqrt{25-y^2}$$
$$x^2=25-y^2$$
$$x^2-25=-y^2$$
$$25-x^2=y^2 \quad \Rightarrow \quad \boxed{y=\pm\sqrt{25-x^2}}$$

We need to chose the positive function here; since the both the inputs and outputs of f are in the interval [0,5], the same must be true of its inverse function.

So, $\boxed{f^{-1}(x)=\sqrt{25-x^2}}$

Notice that this function is its own inverse! In class I once mentioned that this is so whenever the graph of a function is symmetric about the line *y=x*. The graph of f is the part of the circle of radius 5 centered at the origin that lies in Quadrant I, and so indeed has this symmetry.

---

38. The given function $f(x)=(x+6)^2$ is not one-to-one. Find a restricted domain for which the function is one-to-one. Also find the inverse of the function with the restricted domain.

There are two main ways to restrict f's domain to yield a one-to-one function; we can restrict the domain of f to the interval $(-\infty,-6]$, or to the interval $[-6,\infty)$.

Let's begin by choosing the domain of f to be $\boxed{x \ge -6}$.

$$y=(x+6)^2 \quad \rightarrow \quad x=(y+6)^2$$
$$\pm\sqrt{x}=y+6$$
$$-6\pm\sqrt{x}=y \quad \Rightarrow \quad \boxed{y=-6\pm\sqrt{x}}$$

The branch of the graph of f (which we're now finding the inverse of) goes through the point (–4, 4), so the inverse function's graph goes through the point (4, –4). Thus the inverse function that we need has the +:

$$\boxed{f^{-1}(x)=-6+\sqrt{x}} \quad =\sqrt{x}-6$$

---

39. The given function $f(x)=|x+8|$ is not one-to-one. If the domain of this function is restricted so that it is made to be one-to-one, determine the corresponding inverse function and its domain.

There are two main ways to restrict f's domain to yield a one-to-one function; we can restrict the domain of f to the interval $(-\infty,-8]$, or to the interval $[-8,\infty)$.

Let's begin by choosing the domain of f to be $\boxed{x \geq -8}$. Over this interval, $f(x) = x + 8$.

$$y = x + 8 \quad \rightarrow \quad x = y + 8$$

$$x - 8 = y \quad \Rightarrow \quad \boxed{f^{-1}(x) = x - 8}$$

Since the range of our f is all non-negative real numbers, that too is the domain of f^{-1}, $\boxed{[0, \infty)}$.

Remember: " Domain of f " = " Range of f^{-1} " , and

" Domain of f^{-1} " = " Range of f " .