## What is *e* ?

Most simply, $e$ is just a real number, a particular point on the real number line. Like $\pi$, $e$ is also an irrational number (more specifically, they are *transcendental numbers*, numbers that don't occur as roots of polynomials with rational coefficients). Approximately,

$$e = 2.71828182845904523536028747135266249775724709369995957496696762772407663035.$$

This number pops up quite frequently in calculus and higher math. The first place students usually encounter this number is in connection with compound interest. If you invest $P_0$ dollars, earning annual interest rate $r$, compounded yearly (this is called simple interest), the amount you have after $t$ years, $P(t)$, is given by:

*Simple Interest*

$$\boxed{P(t) = P_0\left(1+r\right)^t}$$

(compounded yearly)

Simple interest is calculated only once each year. Lenders almost always use compound interest rather than simple interest, with the compounding of interest done monthly. The function describing the above situation, but compounded monthly rather than yearly, is given by:

*Compound Interest*

$$\boxed{P(t) = P_0\left(1+\frac{r}{12}\right)^{12t}}$$

(compounded monthly)

The reason that $r$ is divided by 12 is that each month you only earn 1/12 of a whole year's interest. The 12 times $t$ is because there are 12 compoundings each year. This can easily be generalized to allow for any number of compoundings per year. For daily compounding, just replace the 12 in the last equation by 365. The function that describes the above situation, but compounded $N$ times per year, is:

*Compound Interest*

$$\boxed{P(t) = P_0\left(1+\frac{r}{N}\right)^{Nt}}$$

(compounded $N$ times per year)

To compound every second, since there are $365 \cdot 24 \cdot 60 \cdot 60 = 31,536,000$ seconds in a year, we would substitute $N = 31{,}536{,}000$ into the above equation. We could even compound every microsecond (one millionth of a second), or picoseconds, or….. If we take the limit as $N$ goes to infinity, we have *continuous compounding of interest*. And this is where $e$ makes its appearance, since

$$\lim_{N\to\infty} P_0\left(1+\frac{r}{N}\right)^{Nt} = P_0 \lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{Nt} = P_0\, e^{rt}$$

The equation for continuous compounding of interest is thus:

*Continuous Compounding*

$$\boxed{P(t) = P_0\, e^{rt}}$$

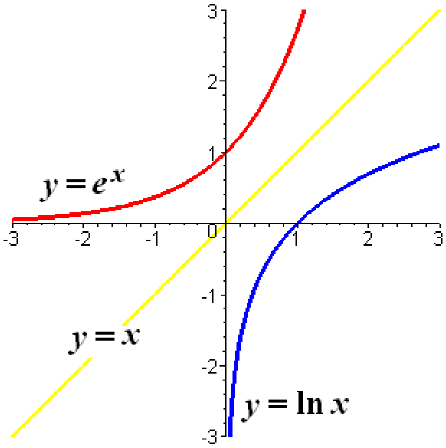
(compounded ∞ times per year)

Amazingly, this same equation can be used to not only solve problems involving continuous compounding of interest, but also applies to population growth and radioactive decay (in which case *r* is negative). In fact, this equation arises in any situation where the rate of change of a quantity is proportional to the quantity itself. Here are a few examples of such situations:

- The more people, the more babies are born in any given time period (with a constant birth rate).
- The more money you invest, the more money you make (with a constant interest rate).
- The more radioactive substance you have, the more radioactive decay is occurring.

The exponential function $\exp(x) = e^x$ is thus quite useful, as is its inverse function, the natural log. The natural log is simply the log to the base *e*, and so $e^{\ln x} = \ln e^x = x$. Here are the graphs of these two functions on the same graph.

Notice how the graphs reflect each other through the line $y = x$:



There are many interesting formulas involving *e*. One that's a consequence of continuous compounding as the limit as the number of compoundings *N* goes to infinity is the equation:

$$\boxed{e = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{N}}$$

Our newfound friend *e* is intimately related to the ordinary counting numbers too, as can be seen in this surprising infinite sum:

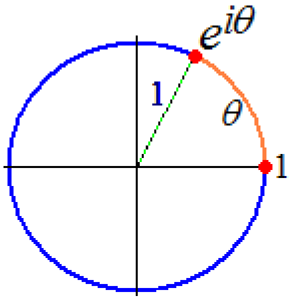
$$\boxed{e = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots}$$

Those exclamation marks indicate *factorials*. ( E.g, $3! = 3 \cdot 2 \cdot 1 = 6$, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$, etc.)

If we allow complex numbers into the picture, we get the amazing Euler identity:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

In the complex plane $e^{i\theta}$ is that point on the unit circle at distance $\theta$ along a counter-clockwise circular arc from the starting point $e^{i\cdot 0} = e^{0} = 1$.



If you let $\theta = \pi$ in Euler's identity, we get the following formula:

$$\boxed{e^{i\pi} = -1}$$

Adding 1 to both sides, we get an wondrous little formula that relates *e*, $\pi$, *i*, 0, and 1:

$$\boxed{e^{i\pi} + 1 = 0}$$

Sometimes the exponential function is defined by a power series (which turns out to be valid not only for all real numbers, but all complex numbers as well, and even for certain *n*-by-*n* matrices):

$$\boxed{e^{x} = \sum_{n=1}^{\infty} \frac{x^{n}}{n!}}$$

From this (and a little advanced calculus), one can deduce all of trigonometry, as well as the algebra of logarithms and exponentials to arbitrary bases. The exponential function has not surprisingly been called the single most important function in all of mathematics.