Chapter 2 Section 6 Lesson
Squares, Square Roots, and Absolute Value

Introduction

This lesson explains squared numbers, the square root, and the idea of absolute value.

Squared Numbers

To **square** a number means to multiply the number by itself.

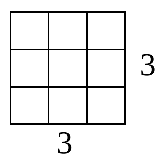
The superscript ^{2} after a number indicates that the number is to be squared. For example, 3^2 means "square the number 3," or "3 squared" for short.

Variables may be squared. For any real number, a:

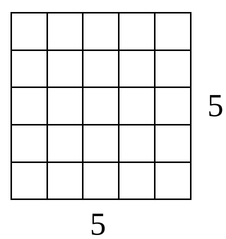
$$a^2 = a \times a$$

It might help to think of the area of a square.

Square the number 3: $3^2 = 3 \times 3 = 9$



Square the number 5: $5^2 = 5 \times 5 = 25$



Not all numerical squares can easily be pictured as geometric squares.

Examples:

Square $\frac{1}{2}$: $\left(\frac{1}{2}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

Note: When we square this fraction, the product is less than the number being squared.

Square (-5): $(-5)^2 = (-5)(-5) = 25$ Here, -5 is squared.

However, note:

$-5^2 = -(5^2) = -(5 \times 5) = -25$ Here, only 5 is squared.

Parentheses must be used if the negative sign is to be included with the number being squared. Why? It's always true that $-a = (-1)a$, so

$$-5^2 = (-1)5^2 = (-1)25 = -25.$$

Extended Example 1

Square $\left(-\frac{3}{4}\right)$

Hint: Multiply the number by itself. The parentheses indicate the negative sign should be included.

Step 1:

$$\left(-\frac{3}{4}\right)^2 = \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)$$

Hint: Multiply the numerators together and the denominators together.

Answer:

$$= \frac{9}{16}$$

The Square Root

Finding the square root of a number means finding a factor that, when multiplied by itself, equals that number.

Example: A square root of 16 is 4 because $4 \times 4 = 16$. The other square root of 16 is -4 because $-4 \times (-4) = 16$.

Each positive number has two square roots, one positive and the other negative. For example $(-5)^2 = (-5)(-5) = 25$, and also $5^2 = 5 \cdot 5 = 25$, so both 5 and -5 are square roots of 25. The positive square root is called the **principal root**. The $\sqrt{\ }$ symbol, called the **radical** or the **square root sign**, indicates the positive square root of the number inside the radical. The expression under the radical sign is called the **radicand**. Numbers such as $1, 4, 9, 16, 25, \ldots$ are called **perfect squares** because they are the squares of integers.

Examples:

$\sqrt{9} = 3$ because $3 \times 3 = 9$

$\sqrt{144} = 12$ because $12 \cdot 12 = 144$

$\sqrt{\frac{16}{25}} = \frac{4}{5}$ because $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$

$\sqrt{0.09} = 0.3$ because $0.3 \times 0.3 = 0.09$

To indicate a negative root, you must put a negative sign in front of the radical.

Examples: $-\sqrt{16} = -4$ $\quad -\sqrt{\frac{9}{64}} = -\frac{3}{8}$

Question : Evaluate $\sqrt{121}$. Explain your answer.

Answer: $\sqrt{121} = 11$ because $11 \times 11 = 121$

Question : Find the principle square root of 81.

Answer: We find that positive number which, when multiplied by itself, equals 81:

$$9 \times 9 = 81 \text{ , so } \sqrt{81} = 9$$

Extended Example 2

Find all the square roots of 169 .

Hint: Find the positive integer that multiplied by itself equals 169 .

Step 1:

$13 \times 13 = 169$, so the principle square root of 169 is $\sqrt{169} = 13$.

Hint: What negative integer, when squared, yields 169 ?

Answer:

$(-13) \times (-13) = 169$

So, the square roots of 169 consist of its principle square root, 13 , along with the negative of its principle square root, -13 .

The Square Root of a Negative Number

The square root of a negative real number cannot be any real number.

To see why this statement is true, let's imagine that $\sqrt{-4} = R$. If so, then $R \times R = -4$. But that's impossible for any real number R , because if R is positive, then $R \times R$ is positive, and so it can't equal -4 ! And, if R is negative, then $R \times R$ is still positive since a negative times a negative is a positive. Either way, it doesn't equal -4 .

If you try to make your calculator take the square root of a negative number, it will probably display something like "ERROR."

Estimating the Square Root of a Non-Perfect Square

Consider the radical $\sqrt{8}$. Since 8 isn't a perfect square, we can't calculate this square root easily. But we do know that 8 is between the perfect squares 4 and 9. This means that the square root of 8 will be between the square root of 4 and the square root of 9. In other words,

$$\sqrt{4} < \sqrt{8} < \sqrt{9}.$$

Taking the square roots of the radicals on each end, we get

$$2 < \sqrt{8} < 3;$$

Therefore, $\sqrt{8}$ is some number between 2 and 3.

You can get better estimates by systematically guessing—we know $\sqrt{8}$ is between 2 and 3. The mid-point between 2 and 3 is 2.5. Squaring 2.5, we get $2.5^2 = 6.25$. Since this number is still smaller than 8, we now know that $\sqrt{8}$ is between 2.5 and 3. We can repeat this process, restricting $\sqrt{8}$ to smaller and smaller intervals until we know $\sqrt{8}$ quite accurately. But, if your calculator has a square root key, then accurate approximations to square roots are just a button push away! My calculator says $\sqrt{8} \cong 2.8284271212475$. The more digits you use, the more accurate your calculations will be, so it's generally best not to round off until the very last step of a calculation.

Extended Example 3

$\sqrt{57}$ is between which two consecutive positive integers? Also, use your calculator to find $\sqrt{57}$ to 5 decimal places and to 3 decimal places.

Hint: What perfect squares that are consecutive integers are smaller, and larger, than 57?

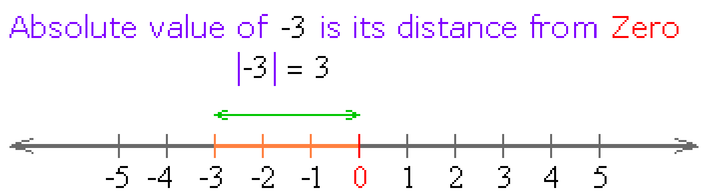
Step 1:
$49 < 57 < 64$.

Hint: Take square roots to find the two consecutive integers that the square root of 57 is sandwiched between.

Answer:
$7 < \sqrt{57} < 8$. $\sqrt{57}$ is between 7 and 8.
Rounded to five decimal places: $\sqrt{57} \cong 7.54983$; to three: $\sqrt{57} \cong 7.550$.

Absolute Value

The **absolute value** of a number is its distance from 0 on the number line. The absolute value of x is written $|x|$. Absolute values are never negative because they refer to only the magnitude, or size, of a number, not its sign. View the animation in the course online.



This image represents an animation that can only be seen in the online course.

Note that $|ab| = |a| \times |b|$. But $|a+b| \neq |a| + |b|$. We can illustrate this by letting $a = 3$ and $b = -7$:

$$|a+b| = |3+(-7)| = |3-7| = |-4| = 4 \quad \text{but} \quad |a| + |b| = |3| + |-7| = 3 + 7 = 10$$

$$\text{So, } |3+(-7)| \neq |3| + |-7|$$

Question: Find the absolute value of -39.

Answer: The absolute value of a number is its distance from zero on the number line.
So, $|-39| = 39$.

The Square Root of a Variable

You can have variables under a radical too, as long as they result in a positive quantity under the radical sign.

You now know that $\sqrt{9} = 3$, but there's more juice to be squeezed from this fruit! Look at that expression in another way: $\sqrt{9} = \sqrt{3 \cdot 3} = \sqrt{3^2} = 3$. Also, $\sqrt{4^2} = 4$, $\sqrt{5^2} = 5$, $\sqrt{\pi^2} = \pi$, and so on. The square root of the square of any non-negative number equals the number itself.

If a represents any non-negative real number, then $\sqrt{a^2} = a$.

The property above is *not* true when a represents a negative number. For example, suppose that $a = -2$. Then $\sqrt{a^2} = \sqrt{(-2)^2} = \sqrt{4} = 2$, which is not equal to a. When the negative number was squared, the negative was lost. But remember that absolute values also result in negative signs being lost. Indeed, if a is a negative number then $\sqrt{a^2} = |a|$. This formula also works if a is positive, since the absolute value of a positive number is that positive number itself.

If a represents any real number, then $\sqrt{a^2} = |a|$.

End of Lesson